問題 4.3　**1.**　次の広義積分を求めよ．

(1)　$\displaystyle\int_0^3 \frac{dx}{\sqrt{3-x}}$

(解答)　$\displaystyle\int \frac{dx}{\sqrt{3-x}} = -2\sqrt{3-x} + C$

$$\int_0^3 \frac{dx}{\sqrt{3-x}} = \lim_{r\to 3-0}\int_0^r \frac{dx}{\sqrt{3-x}} = \lim_{r\to 3-0}\Big[-2\sqrt{3-x}\Big]_0^r = \lim_{r\to 3-0} 2(\sqrt{3}-\sqrt{3-r}) = 2\sqrt{3}$$

(2)　$\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$

(解答)　$\displaystyle\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \frac{1}{\sqrt{u}}\,\frac{du}{dx}dx \qquad \left(u=\sin x,\ \frac{du}{dx}=\cos x\right)$

$$= \int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u} + C = 2\sqrt{\sin x} + C$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{r\to +0}\int_r^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{r\to +0}\Big[2\sqrt{\sin x}\Big]_r^{\frac{\pi}{2}} = \lim_{r\to +0} 2(1-\sqrt{\sin r}) = 2$$

ポイント：$\displaystyle\int g(f(x))f'(x)\,dx = \int g(u)\,\frac{du}{dx}dx \qquad \left(u=f(x),\ \frac{du}{dx}=f'(x)\right)$

$$= \int g(u)\,du$$

問題 4.3　1. 次の広義積分を求めよ.

(3)　$\displaystyle\int_0^\infty xe^{-x}\,dx$

(解答)　$\displaystyle\int xe^{-x}\,dx=\int x\cdot\left(-e^{-x}\right)'\,dx=x\cdot\left(-e^{-x}\right)-\int\left(-e^{-x}\right)dx=-xe^{-x}-e^{-x}+C$

$$\int_0^\infty xe^{-x}\,dx=\lim_{R\to\infty}\int_0^R xe^{-x}\,dx=\lim_{R\to\infty}\Big[-xe^{-x}-e^{-x}\Big]_0^R=\lim_{R\to\infty}1-Re^{-R}-e^{-R}=1$$

ポイント：$\displaystyle\int f(x)g'(x)\,dx=f(x)g(x)-\int f'(x)g(x)\,dx$

α を実数とするとき，$\displaystyle\lim_{R\to\infty}R^\alpha e^{-R}=0$
本問題は，$\alpha=0$，$\alpha=1$ のとき.

(4)　$\displaystyle\int_0^1 x\log x\,dx$

(解答)　$\displaystyle\int x\log x\,dx=\int\log x\cdot\left(\frac{x^2}{2}\right)'\,dx=\log x\cdot\left(\frac{x^2}{2}\right)-\int\frac{1}{x}\cdot\frac{x^2}{2}\,dx=\frac{x^2}{2}\log x-\frac{x^2}{4}+C$

$$\int_0^1 x\log x\,dx=\lim_{r\to+0}\int_r^1 x\log x\,dx=\lim_{r\to+0}\left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]_r^1$$
$$=\lim_{r\to+0}-\frac{1}{4}-\frac{r^2}{2}\log r+\frac{r^2}{4}=-\frac{1}{4}$$

ポイント：$\displaystyle\int f(x)g'(x)\,dx=f(x)g(x)-\int f'(x)g(x)\,dx$

α を正の実数とするとき，$\displaystyle\lim_{r\to+0}r^\alpha\log r=0$
本問題は，$\alpha=2$ のとき.