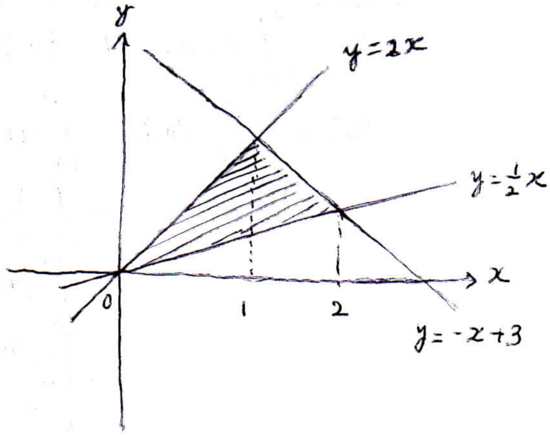
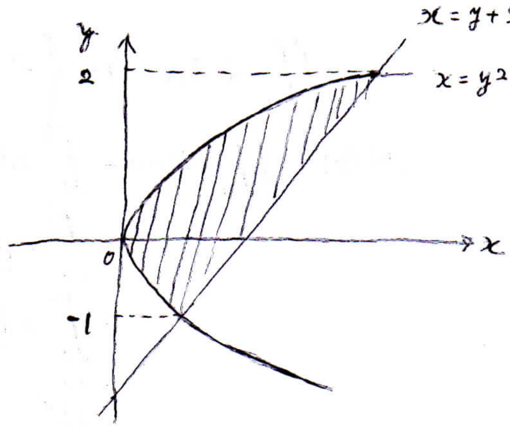
問題 1　　つぎの 2 重積分の値を求めよ．

(5)　$\displaystyle\iint_D dxdy, \quad D=\{y \leqq 2x,\ x \leqq 2y,\ x+y \leqq 3\}.$



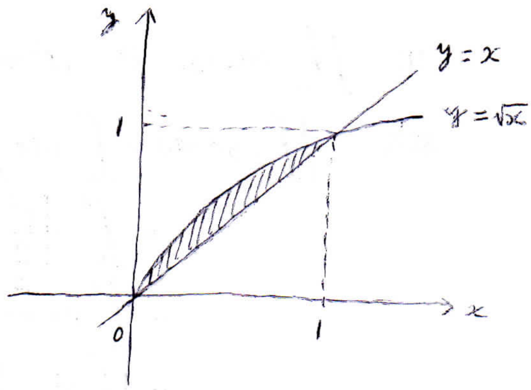
(解答)
$$\iint_D dxdy = \int_0^1 dx \int_{\frac{x}{2}}^{2x} dy + \int_1^2 dx \int_{\frac{x}{2}}^{3-x} dy$$
$$= \int_0^1 \Big[y\Big]_{y=\frac{x}{2}}^{y=2x} dx + \int_1^2 \Big[y\Big]_{y=\frac{x}{2}}^{y=3-x} dx$$
$$= \int_0^1 \frac{3}{2}x dx + \int_1^2 \left(-\frac{3}{2}x+3\right) dx$$
$$= \left[\frac{3}{4}x^2\right]_0^1 + \left[-\frac{4}{3}x^2+3x\right]_1^2 = \frac{3}{2}$$

(6)　$\displaystyle\iint_D y dxdy, \quad D=\{y^2 \leqq x,\ x-2 \leqq y\}.$



(解答)
$$\iint_D y dxdy = \int_{-1}^2 y dy \int_{y^2}^{y+2} dx$$
$$= \int_{-1}^2 y \Big[x\Big]_{x=y+2}^{x=y^2} dy$$
$$= \int_{-1}^2 y(y+2-y^2) dy$$
$$= \left[\frac{1}{3}y^3 + y^2 - \frac{1}{4}y^4\right]_{-1}^2 = \frac{9}{4}$$

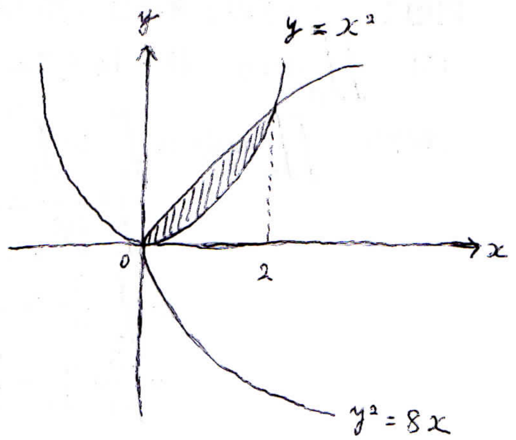
(7)　$\displaystyle\iint_D y dxdy, \quad D=\{0 \leqq x \leqq y \leqq \sqrt{x} \leqq 1\}.$



(解答)
$$\iint_D y dxdy = \int_0^1 dx \int_x^{\sqrt{x}} y dy$$
$$= \int_0^1 \left[\frac{y^2}{2}\right]_{y=x}^{y=\sqrt{x}} dy$$
$$= \int_0^1 \left(\frac{1}{2}x - \frac{1}{2}x^2\right) dy$$
$$= \left[\frac{1}{4}x^2 - \frac{1}{6}x^3\right]_0^1 = \frac{1}{12}$$

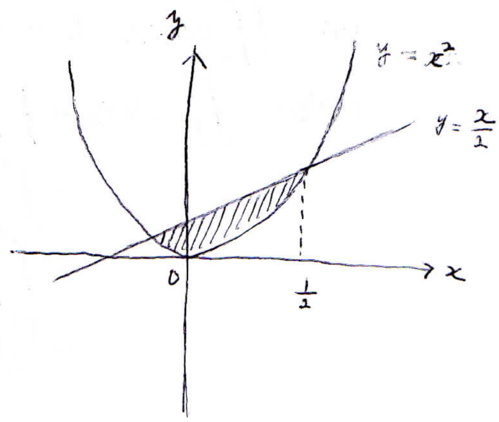
問題 1　つぎの 2 重積分の値を求めよ.

(8) $\displaystyle\iint_D dxdy,\quad D=\{y \geqq x^2,\ y^2 \leqq 8x\}.$



(解答) $$\iint_D dxdy = \int_0^2 dx \int_{x^2}^{2\sqrt{2x}} dy = \int_0^2 \Big[y\Big]_{y=x^2}^{y=2\sqrt{2x}} dx = \int_0^2 (2\sqrt{2x}-x^2)dx = \left[\frac{4\sqrt{2}}{3}x\sqrt{x}-\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

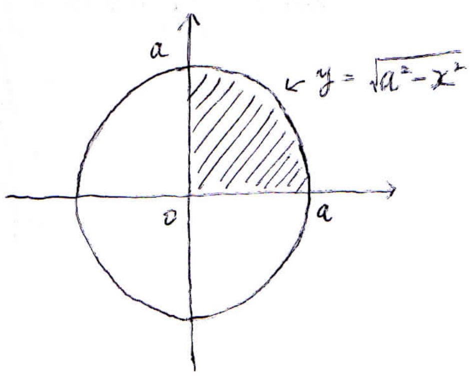
(9) $\displaystyle\iint_D xdxdy,\quad D=\left\{x^2 \leqq y \leqq \frac{x}{2},\ 0 \leqq x \leqq \frac{1}{2}\right\}.$



(解答) $$\iint_D xdxdy = \int_0^{\frac{1}{2}} xdx \int_{x^2}^{\frac{x}{2}} dy = \int_0^{\frac{1}{2}} x\Big[y\Big]_{y=x^2}^{y=\frac{x}{2}} dx = \int_0^{\frac{1}{2}} x\left(\frac{x}{2}-x^2\right)dx = \left[\frac{1}{6}x^3-\frac{1}{4}x^4\right]_0^{\frac{1}{2}} = \frac{1}{192}$$

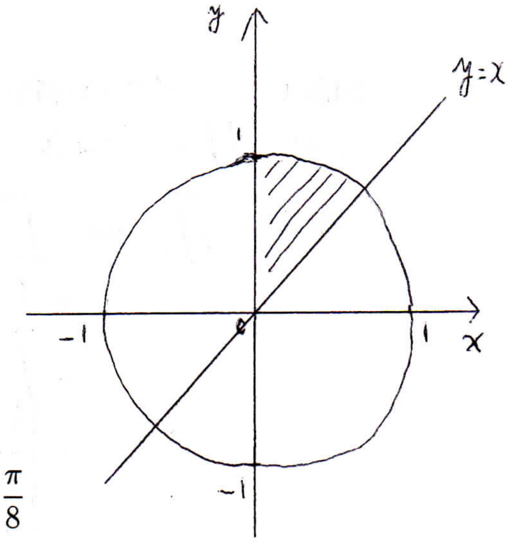
(10) $\displaystyle\iint_D xydxdy,\quad D=\{x^2+y^2 \leqq a^2,\ x \geqq 0,\ y \geqq 0\}\quad (a>0).$

(解答) $$\iint_D xydxdy = \int_0^a xdx \int_0^{\sqrt{a^2-x^2}} ydy = \int_0^a x\left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{a^2-x^2}} dx = \int_0^a x\cdot\frac{a^2-x^2}{2}dx = \left[\frac{a^2}{4}x^2-\frac{1}{8}x^4\right]_0^a = \frac{a^4}{8}$$



問題 1　つぎの 2 重積分の値を求めよ．

(11)　$\displaystyle\iint_D dxdy,\quad D=\{x^2+y^2 \leqq 1,\ 0 \leqq x \leqq y\}.$



(解答)

$$\iint_D dxdy = \int_0^{\frac{1}{\sqrt{2}}} dx \int_x^{\sqrt{1-x^2}} dy$$
$$= \int_0^{\frac{1}{\sqrt{2}}} \Big[y\Big]_{y=x}^{y=\sqrt{1-x^2}} dx$$
$$= \int_0^{\frac{1}{\sqrt{2}}} (\sqrt{1-x^2}-x)dx$$
$$= \left[\frac{1}{2}\left(x\sqrt{1-x^2}+\sin^{-1}x\right)-\frac{x^2}{2}\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{8}$$

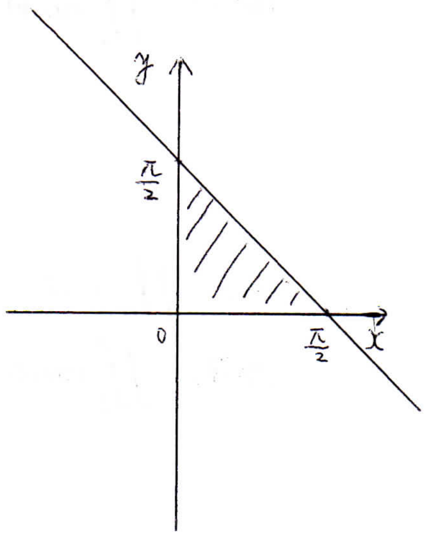
(12)　$\displaystyle\iint_D ydxdy,\quad D=\{x^2+y^2 \leqq 1,\ 0 \leqq x \leqq y\}.$

(解答)

$$\iint_D ydxdy = \int_0^{\frac{1}{\sqrt{2}}} dx \int_x^{\sqrt{1-x^2}} ydy$$
$$= \int_0^{\frac{1}{\sqrt{2}}} \left[\frac{y^2}{2}\right]_{y=x}^{y=\sqrt{1-x^2}} dx$$
$$= \int_0^{\frac{1}{\sqrt{2}}} \left(\frac{1-x^2}{2}-\frac{x^2}{2}\right) dx$$
$$= \left[\frac{1}{2}x-\frac{1}{3}x^3\right]_0^{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{6}$$

図は(11)と同じ

(13)　$\displaystyle\iint_D \sin(x+y)dxdy,\quad D=\left\{x+y \leqq \frac{\pi}{2},\ x \geqq 0,\ y \geqq 0\right\}.$



(解答)

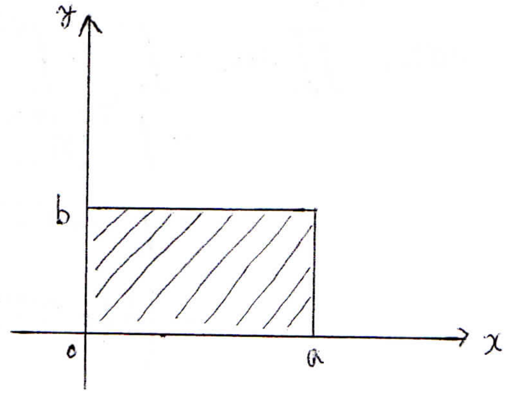
$$\iint_D \sin(x+y)dxdy = \int_0^{\frac{\pi}{2}} dx \int_0^{\frac{\pi}{2}-x} \sin(x+y)dy$$
$$= \int_0^{\frac{\pi}{2}} \Big[-\cos(x+y)\Big]_{y=0}^{y=\frac{\pi}{2}-x} dx$$
$$= \int_0^{\frac{\pi}{2}} \cos x dx$$
$$= \Big[\sin x\Big]_0^{\frac{\pi}{2}} = 1$$

問題 1　　つぎの 2 重積分の値を求めよ．

(14)　$\displaystyle\iint_D e^{2x+3y}dxdy,\quad D=\{0\leqq x\leqq a,\ 0\leqq y\leqq b\}\quad (a>0,\ b>0).$

(解答)

$$\iint_D dxdy=\int_0^a dx\int_0^b e^{2x+3y}dy$$
$$=\int_0^a \left[\frac{1}{3}e^{2x+3y}\right]_{y=0}^{y=b}dx$$
$$=\int_0^a \frac{1}{3}(e^{2x+3b}-e^{2x})dx$$
$$=\left[\frac{1}{6}(e^{2x+3b}-e^{2x})\right]_0^a$$
$$=\frac{1}{6}(e^{2a+3b}-e^{2a}-e^{3b}+1)$$



(15)　$\displaystyle\iint_D rdrd\theta,\quad D=\{0\leqq r\leqq a,\ \alpha\leqq\theta\leqq\beta\}\quad (a>0,\ 0\leqq\alpha<\beta\leqq 2\pi).$

(解答)

$$\iint_D rdrd\theta=\int_\alpha^\beta d\theta\int_0^a rdr=\int_\alpha^\beta\left[\frac{r^2}{2}\right]_{r=0}^{r=a}d\theta$$
$$=\int_\alpha^\beta\frac{a^2}{2}d\theta=\left[\frac{a^2\theta}{2}\right]_\alpha^\beta=\frac{a^2(\beta-\alpha)}{2}$$

(16)　$\displaystyle\iint_D rdrd\theta,\quad D=\{0\leqq r\leqq a\sin\theta,\ 0\leqq\theta\leqq\pi\}\quad (a>0).$

(解答)

$$\iint_D rdrd\theta=\int_0^\pi d\theta\int_0^{a\sin\theta}rdr=\int_0^\pi\left[\frac{r^2}{2}\right]_{r=0}^{r=a\sin\theta}d\theta$$
$$=\int_0^\pi\frac{a^2\sin^2\theta}{2}d\theta=\int_0^\pi\frac{a^2(1-\cos 2\theta)}{4}d\theta$$
$$=\left[\frac{a^2\theta}{4}-\frac{a^2\sin 2\theta}{8}\right]_0^\pi=\frac{\pi a^2}{4}$$

(17)　$\displaystyle\iint_D rdrd\theta,\quad D=\{0\leqq r\leqq 2a(1+\cos\theta),\ 0\leqq\theta\leqq 2\pi\}\quad (a>0).$

(解答)

$$\iint_D rdrd\theta=\int_0^{2\pi}d\theta\int_0^{2a(1+\cos\theta)}rdr=\int_0^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=2a(1+\cos\theta)}d\theta$$
$$=\int_0^{\pi}2a^2(1+\cos\theta)^2d\theta=\int_0^{2\pi}a^2(3+2\cos\theta+\cos 2\theta)d\theta$$
$$=\left[a^2\left(3\theta+2\sin\theta+\frac{\sin 2\theta}{2}\right)\right]_0^{2\pi}=6\pi a^2$$

(16) と (17) のポイント：
$$\begin{cases}\sin^2\theta=\dfrac{1-\cos 2\theta}{2}\\[2mm] \cos^2\theta=\dfrac{1+\cos 2\theta}{2}\end{cases}$$